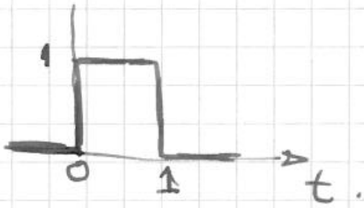
E 2.5 Signals and Systems.

Tutorial Sheet 4 Solutions

1. a) $f(t) = u(t) - u(t-1)$



$$F(s) = \int_0^1 e^{-st}\,dt$$

$$= -\frac{e^{-st}}{s}\Big|_0^1$$

$$= -\frac{1}{s}\left(e^{-s} - 1\right)$$

$$= \frac{1}{s}\left(1 - e^{-s}\right)$$

b) $f(t) = t e^{-t} u(t)$

$$F(s) = \int_0^\infty t e^{-t} e^{-st}\,dt = \int_0^\infty t e^{-(s+1)t}\,dt$$

Use integration by parts $\Longrightarrow$

$f(t) = t$

$g'(t) = e^{-(s+1)t}$

$$= -\frac{e^{-(s+1)t}}{(s+1)^2}\left[-(s+1)t - 1\right]\Big|_0^\infty$$

In order to guarantee convergence, we need $e^{-(s+1)t} \to 0$ as $t \to \infty$, or $\mathrm{Re}\,(s+1) > 0$.

Then

$$F(s) = \underbrace{\frac{e^{-(s+1)t}}{(s+1)^2}(s+1)t\Big|_0^\infty}_{0} + \frac{e^{-(s+1)t}}{(s+1)^2}\Big|_0^\infty$$

$$= \frac{1}{(s+1)^2}$$

(Note! This question is actually quite difficult!).

c) $f(t) = t\cos\omega_0 t\, u(t)$

$$F(s) = \int_0^\infty t\cos\omega_0 t\, e^{-st}\,dt$$

$$= \frac{1}{2}\left\{\int_0^\infty \left[t e^{(j\omega_0 - s)t} + t e^{-(j\omega_0 + s)t}\right] dt\right\}$$

$$= \frac{1}{2}\left[\frac{1}{(s-j\omega_0)^2} + \frac{1}{(s+j\omega_0)^2}\right] \qquad \mathrm{Re}\,(s) > 0$$

$$= \frac{s^2 - \omega_0^2}{(s^2+\omega_0^2)^2} \quad //$$

d) $f(t) = e^{-2t}\cos(5t+\theta)\, u(t)$

$$= \frac{1}{2}\left[e^{-2t + j(5t+\theta)} + e^{-2t - j(5t+\theta)}\right] u(t)$$

$$= \left\{\frac{1}{2}e^{j\theta} e^{-(2-j5)t} + \frac{1}{2}e^{-j\theta} e^{-(2+j5)t}\right] u(t)$$

$$\therefore F(s) = \frac{1}{2}e^{j\theta}\left(\frac{1}{s+2-j5}\right) + \frac{1}{2}e^{-j\theta}\left(\frac{1}{s+2+j5}\right)$$

$$= \frac{1}{2} \times \frac{1}{(s^2+4s+29)}\left[(s+2+j5)e^{j\theta} + (s+2-5j)e^{-j\theta}\right]$$

$$= \frac{(s+2)\cos\theta - 5\sin\theta}{s^2+4s+29} \quad //$$

(Also quite hard!)

③

Integration by parts

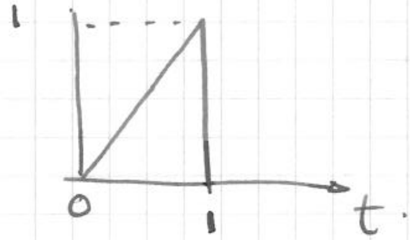
$$\int_a^b f(t)\,g'(t)\,dt$$

$$= f(t)\,g(t)\Big|_a^b - \int_a^b f'(t)\,g(t)\,dt$$

Let $g'(t) = e^{-st}$

$g(t) = -\frac{1}{s}e^{-st}$ $\quad f'(t) = 1$

2 a)



$$F(s) = \int_0^1 t\,e^{-st}\,dt$$

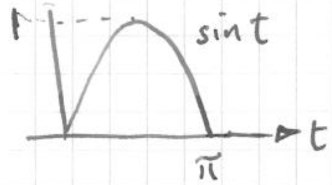
$$= -\frac{t}{s}e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt$$

$$= -\frac{t}{s}e^{-st}\Big|_0^1 - \frac{1}{s^2}e^{-st}\Big|_0^1$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s}$$

$$= \frac{1}{s^2}\left(1 - e^{-s} - se^{-s}\right) \quad //$$

b)

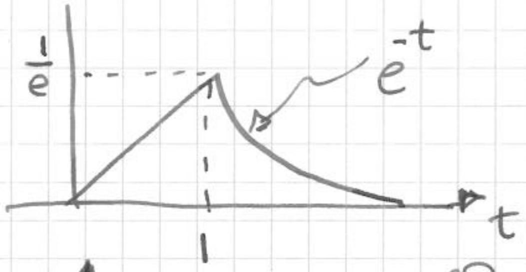


$$F(s) = \int_0^{\pi} \sin t\, e^{-st}\,dt$$

$$= \frac{e^{-st}}{s^2+1}\left(-s\sin t - \cos t\right)\Big|_0^{\pi}$$

$$= \frac{1+e^{-\pi s}}{s^2+1} \quad //$$

c)



$$F(s) = \int_0^1 \frac{t}{e} e^{-st}\,dt + \int_1^\infty e^{-t} e^{-st}\,dt$$

$$= \frac{1}{e}\int t e^{-st}\,dt + \int_1^\infty e^{-(s+1)t}\,dt$$

$$= \frac{e^{-st}}{es}(-st-1)\Big|_0^1 - \frac{1}{s+1} e^{-(s+1)t}\Big|_1^\infty$$

$$= \underbrace{\frac{1}{es^2}\left(1-e^{-s}-se^{-s}\right)}_{\text{similar to Q2a)}} + \frac{1}{s+1} e^{-(s+1)}$$

3 a) $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{s^2+5s+6}\right\}$

$$\frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$\therefore\ f(t) = \left(e^{-2t} + e^{-3t}\right) u(t) \quad //$$

b) $F(s) = \dfrac{3s+5}{s^2+4s+13}$

Use Table pair 10 c)

$A = 3,\quad B = 5,\quad a = 2,\quad c = 13.$

$$\therefore\ b = \sqrt{c-a^2} = \sqrt{13-4} = 3$$

$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c-a^2}} = \sqrt{\frac{117+25-60}{13-4}}$$

$$= 3.018$$

$$\theta = \tan^{-1}\left(\frac{Aa-B}{A\sqrt{c-a^2}}\right) = \tan^{-1}\left(\frac{1}{9}\right) = 6.34^\circ$$

$$\therefore\ f(t) = 3.018\, e^{-2t} \cos(3t + 6.34^\circ)\, u(t) \quad //$$

3 c) $F(s) = \frac{(s+1)^2}{s^2 - s - 6} = \frac{(s+1)^2}{(s+2)(s-3)}$

Since the order of numerator = order of denominator, this is an improper fraction.

An example is given in Lecture 6 slide 12.

Using method provided in lecture,

$$F(s) = 1 + \frac{a}{s+2} + \frac{b}{s-3} = 1 - \frac{0.2}{s+2} + \frac{3.2}{s-3}$$

This is the coefficient of the s^2 term in numerator.

$$\therefore \quad f(t) = \delta(t) + \left(3.2e^{3t} - 0.2e^{-2t}\right)u(t)$$

d) $F(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{-1}{s+1} + \frac{As+B}{s^2+2s+2}$

Normal partial fraction trick.

Multiply both sides by s and let $s \to \infty$. This gives:

$$\left.\frac{2s^2+s}{(s+1)(s^2+2s+2)}\right|_{s\to\infty} = \left. -\frac{s}{s+1} + \frac{As^2+Bs}{s^2+2s+2}\right|_{s\to\infty}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$0 = -1 + A \Rightarrow A = 1$$

Set $s=0$ on both sides :-

$$\frac{1}{2} = -1 + \frac{B}{2} \Rightarrow B = 3$$

$$\therefore \ F(s) = -\frac{1}{s+1} + \frac{s+3}{s^2+2s+2}$$

Use Table of Laplace Transform: $f(t) = \left[-e^{-t} + \sqrt{5}\,e^{-t}\cos(t - 63.4°)\right]u(t)$

4 a) $f(t) = u(t) - u(t-1)$

$$\therefore \quad F(s) = \mathcal{L}[u(t)] - \mathcal{L}[u(t-1)]$$

$$= \frac{1}{s} - e^{-s}\frac{1}{s} = \frac{1}{s}(1 - e^{-s})$$

Important: Compare this solution with that of Q1 a), this is much easier.

b) $f(t) = e^{-(t-\tau)}u(t) = e^{\tau}\, e^{-t}u(t)$.

$$\therefore \quad F(s) = e^{\tau}\frac{1}{s+1}$$

c) $f(t) = e^{-t}\,u(t-\tau) = e^{-\tau}\, e^{-(t-\tau)}\,u(t-\tau)$

Note that $e^{-(t-\tau)}u(t-\tau)$ is $e^{-t}u(t)$ delayed by τ.

$$\therefore \quad F(s) = e^{-\tau}\left(\frac{1}{s+1}\right)e^{-s\tau} = \left(\frac{1}{s+1}\right)e^{-(s+1)\tau}$$

d) $f(t) = \sin \omega_0(t-\tau)\, u(t-\tau)$.

This is $\sin \omega_0 t$ delayed by τ.

$$\therefore \quad F(s) = \left(\frac{\omega_0}{s^2+\omega_0^2}\right)e^{-s\tau}$$

e) $f(t) = \sin \omega_0 (t-\tau)\, u(t)$

$$= [\sin \omega_0 t \underbrace{\cos \omega_0 \tau}_{\text{Constant}} - \cos \omega_0 t \underbrace{\sin \omega_0 \tau}_{\text{constant}}]\, u(t)$$

$$\therefore \quad F(s) = \left(\frac{\omega_0}{s^2+\omega_0^2}\right)\cos \omega_0 \tau - \left(\frac{s}{s^2+\omega_0^2}\right)\sin \omega_0 \tau$$

$$= \frac{\omega_0 \cos \omega_0 \tau - s \sin \omega_0 \tau}{s^2+\omega_0^2}$$

5. $$F(s) = \frac{(2s+5)e^{-2s}}{s^2+5s+6}$$

$$= \hat{F}(s) \cdot e^{-2s}, \text{ where } \hat{F}(s) = \frac{2s+5}{s^2+5s+6}$$

Let us now find $\mathcal{L}^{-1}\{\hat{F}(s)\}$.

$$\hat{F}(s) = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3}$$

$$\therefore \hat{f}(t) = \left(e^{-2t} + e^{-3t}\right) u(t).$$

Using time-shifting property,

$$f(t) = \hat{f}(t-2)$$

$$\therefore f(t) = \left[e^{-2(t-2)} + e^{-3(t-2)}\right] u(t-2)$$

6. This is hard question. The key is
a) to recognise that:

$$g(t) = f(t) + f(t - T_0) + f(t - 2T_0) + \cdots\cdots$$

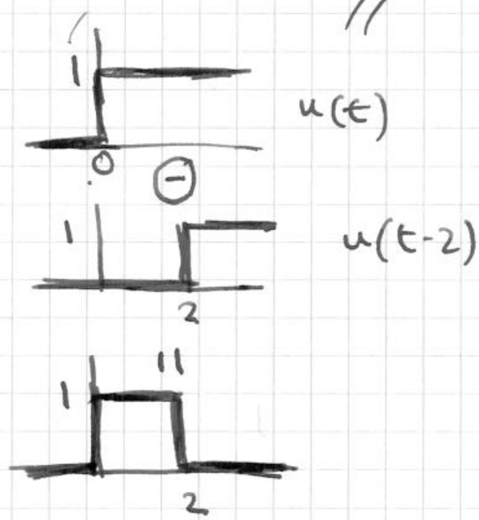
$$\therefore\ G(s) = F(s) + F(s)e^{-sT_0} + F(s)e^{-2sT_0} + \cdots\cdots$$

$$= F(s)\left[1 + e^{-sT_0} + e^{-2sT_0} + \cdots\cdots\right]$$

$$= \frac{F(s)}{1 - e^{-sT_0}} \quad \text{for } |e^{-sT_0}| < 1$$

or $\mathrm{Re}\ s > 0$ //

b) $$f(t) = u(t) - u(t-2)$$



$$F(s) = \frac{1}{s}\left(1 - e^{-2s}\right)$$

$\therefore$ From a)

$$G(s) = \frac{F(s)}{(1 - e^{-8s})}$$

$$= \frac{1}{s}\,\frac{(1 - e^{-2s})}{(1 - e^{-8s})}$$

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